

Dissipative structures in an optomechanical cavity model with a microstructured oscillating end mirror

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We theoretically consider the possibility of generating nonlinear transverse patterns in an optomechanical system in which the mechanical element is a microstructured oscillating end-mirror consisting in an array of weakly-coupled micromirrors. While in the limit of large number of micromirrors we naturally recover the continuous model that we recently studied in [Ruiz-Rivas et al., Phys. Rev. A **93**, 033850 (2015)], we pay special numerical attention to the opposite limit, showing that the structures predicted with the continuous model can be observed for a number of micromirrors as low as ten or less. This opens new venues for experimental approaches to the subject.

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I. INTRODUCTION

Dissipative structures are long-range ordered structures that spontaneously form in extended nonlinear dissipative systems when brought apart from thermal equilibrium [1]. Widespread in nature, in nonlinear optics such patterns appear in the distribution of the light field across the plane transverse to its propagation direction, and have been theoretically studied and experimentally observed in many different nonlinear optical cavities [2, 3]. Part of the interest of this research program lies in the potential for optical information storage and processing of a particular type of pattern, cavity solitons, which are localized structures that can be individually written, erased, and even moved without affecting neighbouring structures [4–6].

In a recent paper [7] we theoretically analyzed the possibility of using the nonlinear coupling between the cavity field and a deformable mechanical element to generate transverse patterns. We demonstrated the feasibility of pattern formation in such optomechanical (OM) cavities, but only provided that a certain condition, not needed in other nonlinear optical cavities, is fulfilled: the mechanical element must possess a sufficiently homogeneous mode, what is not easily obtained in current implementations. Along these lines, below we propose a special architecture for the OM cavity that allows for pattern formation, hence providing an alternative to our previous proposal in [7].

OM cavities [8] are conceptually simple systems, consisting of an optical resonator with mechanical degrees of freedom that couple to the light oscillating inside it. The coupling appears either through radiation pressure (e.g., when the mechanical degree of freedom corresponds to the oscillation of a perfectly reflecting cavity mirror) or through dispersive effects (e.g., when the mechanical degrees of freedom correspond to the local displacement of a partially transmitting membrane). These systems are receiving intense and continued attention mainly in the context of modern quantum technologies, where phenom-

ena such as cooling, induced transparency, and squeezing have been demonstrated along the last decade [8]. Generating dissipative structures in OM cavities could provide them with new capabilities, and it could also lead to a pattern forming system in which quantum fluctuations play an important role: as OM cavities have demonstrated their ability to work within the quantum regime, they open the possibility of studying quantum dissipative structures under the strong influence of quantum fluctuations. In the past some exciting phenomena concerning quantum fluctuations in dissipative structures were predicted for optical parametric oscillators [9–13], however those models are far from realistic experimental implementations [14]. Contrarily, OM cavities have demonstrated the feasibility of the simplest models as well as a large versatility because of the variety of possible platforms, materials and designs [8]. As a first step towards understanding this fully quantum picture, it is important to characterize the conditions required for pattern formation in OM cavities.

As mentioned above, one of the conclusions in [7] was that the existence of an homogeneous transverse mode is needed in order for patterns to appear, i.e., the mechanical element (be it an end mirror or an intracavity membrane) must be allowed to oscillate back and forth without loosing its flatness when homogeneously illuminated. We suggested a way for implementing such a device through an quasi-one-dimensional membrane, that is, a membrane clamped by a large aspect ratio frame, such that only one or a few transverse modes are excited along the short direction y . Under these conditions the system naturally develops a one-dimensional homogeneous mode along the long direction, which becomes unstable through pattern forming instabilities under appropriate conditions (parameter settings).

Here we investigate a conceptually different possibility by considering an OM cavity with an oscillating microstructured end-mirror consisting of an array of N weakly-coupled micromirrors, see Fig. 1 for a sketch. Below we show that in the limit of large N one recovers the con-

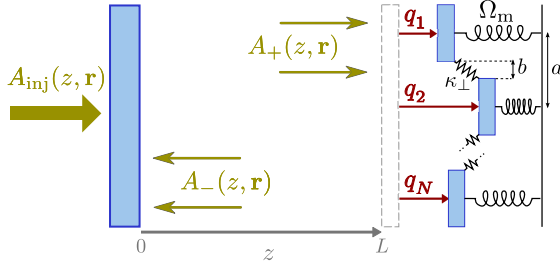


FIG. 1: Sketch of the optomechanical cavity. The moving mirror is formed by a number of micromirrors that are weakly coupled.

tinuous linear-coupling model in [7], and hence all the predictions in that work apply when the number of elements in the array is very large. The interesting point is that one can consider also the limit of small N and study the transition from the continuous to the discrete limit. Among the interesting numerical results that we show below it is remarkable the fact that a discrete analog of the continuous limit cavity solitons can be observed with a relatively small number of coupled micro-mirrors, say $N \lesssim 10$.

II. MODEL

Consider an optical cavity with large-area mirrors one of which is plane, partially transmitting, and immune to radiation pressure because of its stiffness and mass, while the other is a perfectly reflecting array of weakly coupled micro-mirrors (see Fig. 1). The field injected in the cavity through the coupling mirror is assumed to be a paraxial, coherent beam

$$E_{\text{inj}}(z, \mathbf{r}, t) = i\mathcal{V}A_{\text{inj}}(z, \mathbf{r}, t)e^{i(k_L z - \omega_L t)} + \text{c.c.}, \quad (1)$$

where $\mathbf{r} = (x, y)$ denotes the position in the plane transverse to the cavity axis (z -axis), and \mathcal{V} is a constant having the dimensions of voltage, which is usually chosen as $\mathcal{V} = \sqrt{\hbar\omega_c/4\epsilon_0 L}$ in order to make contact with quantum optics (see Appendix A), ω_c being the frequency of the longitudinal cavity mode closest to the injected frequency ω_L with corresponding wave vector $k_L = \omega_L/c$.

The generic intracavity field $E(z, \mathbf{r}, t)$ can be written as

$$E(z, \mathbf{r}, t) = i\mathcal{V}(A_+e^{ik_L z} + A_-e^{-ik_L z})e^{-i\omega_L t} + \text{c.c.}, \quad (2)$$

which is the superposition of two waves with slowly varying complex amplitudes $A_{\pm}(z, \mathbf{r}, t)$, propagating along the positive (A_+) and negative (A_-) z direction. With similar assumptions as those in [7, 15] (see Appendix B for the derivation), the field $A_+(z=L, \mathbf{r}, t)$ at the microstructured mirror's surface, which we denote by $A(\mathbf{r}, t)$, has the following evolution equation

$$\partial_t A = \gamma_c \left(-1 + i\Delta + il_c^2 \nabla_{\perp}^2 + i\frac{4k_L}{T} Q \right) A + \gamma_c \mathcal{E}. \quad (3)$$

Here T is the transmissivity of the fixed mirror, $\gamma_c = cT/4L$ the cavity damping rate, $\Delta = (\omega_L - \omega_c)/\gamma_c$ the dimensionless detuning parameter, $l_c^2 = 2L/k_L T$ the square of the diffraction length, $\nabla_{\perp}^2 = \partial_x^2 + \partial_y^2$ the transverse Laplacian, and $\mathcal{E}(\mathbf{r}, t) = 2T^{-1/2}A_{\text{inj}}(L, \mathbf{r}, t + t_c)$ a scaled version of the injection field amplitude ($t_c = 2L/c$ is the cavity round-trip time). In Eq. (3) we have introduced a field $Q(\mathbf{r}, t)$ that measures the local displacement of the flexible mirror perpendicular to its flat state (hence $Q = 0$ at rest) and next we derive its equation of motion.

We describe the displacement of the flexible mirror in terms of the individual displacements $\{q_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{N}^2}$ of its constituent micro-mirrors, labelled by a double index $\mathbf{j} = (j_x, j_y)$ in a 2D configuration, as

$$Q(\mathbf{r}, t) = \sum_{\mathbf{j}} q_{\mathbf{j}}(t) w_{\mathbf{j}}(\mathbf{r}), \quad (4)$$

where $w_{\mathbf{j}}(\mathbf{r})$ is a function which equals 1 when \mathbf{r} is on the surface of micro-mirror \mathbf{j} and is zero otherwise. In the following we assume for simplicity that the micro-mirrors are much larger than the separation between them, that is, $a \gg b$ in Fig. 1. Each of these displacements is assumed to satisfy the equation of motion of a damped and forced harmonic oscillator, the force acting on mirror \mathbf{j} having two contributions, $F_{\mathbf{j}} = F_{\mathbf{j}}^{(\text{RP})} + F_{\mathbf{j}}^{(\perp)}$, respectively coming from radiation pressure and from the coupling to neighbouring mirrors. The first contribution is readily obtained by integrating the radiation pressure (22) over the surface $\mathcal{S}_{\mathbf{j}}$ of the corresponding micro-mirror

$$F_{\mathbf{j}}^{(\text{RP})} = \frac{2\hbar k_c}{t_c} \int_{\mathcal{S}_{\mathbf{j}}} d^2\mathbf{r} |A(\mathbf{r}, t)|^2, \quad (5)$$

where $k_c = \omega_c/c$. As for the force coming from the coupling to neighbouring mirrors, we assume that it originates from a potential that harmonically couples neighbours as $V_{\mathbf{j}}^{\perp} = \kappa_{\perp} \sum_{\langle \mathbf{l} \rangle_{\mathbf{j}}} (q_{\mathbf{l}} - q_{\mathbf{j}})^2/2$, where $\langle \mathbf{l} \rangle_{\mathbf{j}}$ means that the sum is performed over nearest neighbours, hence the corresponding force reads

$$F_{\mathbf{j}}^{(\perp)} = -\frac{\partial V_{\mathbf{j}}^{\perp}}{\partial q_{\mathbf{j}}} = \kappa_{\perp} \sum_{\langle \mathbf{l} \rangle_{\mathbf{j}}} (q_{\mathbf{l}} - q_{\mathbf{j}}), \quad (6)$$

which for a generic micro-mirror (not at the boundary of the flexible mirror) takes the explicit form

$$F_{\mathbf{j}}^{\perp} = \kappa_{\perp} (q_{\mathbf{j}+\mathbf{x}} + q_{\mathbf{j}-\mathbf{x}} + q_{\mathbf{j}+\mathbf{y}} + q_{\mathbf{j}-\mathbf{y}} - 4q_{\mathbf{j}}). \quad (7)$$

Note that \mathbf{x} and \mathbf{y} are the unit vectors along the x and y directions, respectively. Putting everything together we get

$$\begin{aligned} \ddot{q}_{\mathbf{j}} + \gamma_m \dot{q}_{\mathbf{j}} + \Omega_m^2 q_{\mathbf{j}} &= \frac{\kappa_{\perp}}{m} \sum_{\langle \mathbf{l} \rangle_{\mathbf{j}}} (q_{\mathbf{l}} - q_{\mathbf{j}}) \\ &+ \frac{2\hbar k_c}{t_c m} \int_{\mathcal{S}_{\mathbf{j}}} d^2\mathbf{r} |A(\mathbf{r}, t)|^2, \end{aligned} \quad (8)$$

with γ_m , Ω_m and m the damping rate, oscillation frequency, and mass of the micromirrors, respectively. Eq. (8) together with the optical field Eq.(3) and definition (4) form the equations of our model.

III. CONTINUOUS LIMIT

Now we consider the limit in which the number of micromirrors per unit length tends to infinity while keeping a finite mass density and sound speed. We first write the displacements as a function of the continuous mechanical field Q as

$$q_j = \int_{\mathbb{R}^2} \frac{d^2\mathbf{r}}{a^2} Q(\mathbf{r}) w_j(\mathbf{r}). \quad (9)$$

Next, using the immediate properties

$$\int_{S_j} d^2\mathbf{r} |A(\mathbf{r}, t)|^2 = \int_{\mathbb{R}^2} d^2\mathbf{r} |A(\mathbf{r}, t)|^2 w_j(\mathbf{r}), \quad (10a)$$

$$\int_{\mathbb{R}^2} d^2\mathbf{r} Q(\mathbf{r}) w_{j\pm\mathbf{u}}(\mathbf{r}) = \int_{\mathbb{R}^2} d^2\mathbf{r} Q(\mathbf{r} \mp a\mathbf{u}) w_j(\mathbf{r}), \quad (10b)$$

with $\mathbf{u} = \mathbf{x}, \mathbf{y}$, one gets from the equation of motion of a generic displacement q_j , Eq. (8),

$$\begin{aligned} \partial_t^2 Q(\mathbf{r}) + \gamma_m \partial_t Q(\mathbf{r}) + \Omega_m^2 Q(\mathbf{r}) &= \frac{2\hbar k_c a^2}{t_c m} |A(\mathbf{r})|^2 \\ &+ \Omega_\perp^2 [Q(\mathbf{r} + a\mathbf{x}) + Q(\mathbf{r} - a\mathbf{x}) \\ &+ Q(\mathbf{r} + a\mathbf{y}) + Q(\mathbf{r} - a\mathbf{y}) - 4Q(\mathbf{r})], \end{aligned} \quad (11)$$

with $\Omega_\perp = \sqrt{\kappa_\perp/m}$. Note that we are considering a micromirror that is not at the boundary of the flexible mirror (as in the continuous limit the fields extend up to infinity), and hence have used Eq. (7).

The last step consists in taking the limit $a \rightarrow 0$, but keeping finite both the speed at which transverse perturbations propagate in the flexible mirror $v = a\Omega_\perp$ and its surface mass density $\sigma = m/a^2$. In this limit we can approximate

$$Q(\mathbf{r} \pm a\mathbf{u}) \simeq Q(\mathbf{r}) \pm a\partial_u Q(\mathbf{r}) + a^2\partial_u^2 Q(\mathbf{r})/2, \quad (12)$$

which when used in (11) leads to

$$\partial_t^2 Q + \gamma_m \partial_t Q + (\Omega_m^2 - v^2 \nabla_\perp^2) Q = \frac{2\hbar k_c}{t_c \sigma} |A|^2. \quad (13)$$

This is the same equation of motion we derived in the linear-coupling model of Ref. [7].

IV. NUMERICAL SIMULATIONS

From the previous derivation of the continuous model we can conclude that, at least in the limit of a very large number of micromirrors, the system we are proposing is

equivalent to that studied in [7]. This implies that all the results that we obtained in [7] for the linear coupling model apply in this limit, both the analytical (concerning homogeneous steady states and their stability properties) and the numerical ones (types of patterns, generalized bistability, temporal dynamics, etc.). Of course one must wonder how large must the density of micro-mirrors be for the results of the continuous model to still apply, as well as how do the results change when departing from such continuous limit.

We have performed extensive numerical simulations of both the discrete and continuous models, and Figs. 2 and 3 summarize our main findings. For simplicity, we have restricted the simulations to one dimension, but similar conclusions are drawn in 2D. We have numerically simulated the continuous model by using the usual split-step method, which at any time step provides an approximation of the fields at certain space points. The same method can be applied to the discrete model, and in particular, we take M spatial points for the optical field at every micromirror, denoting by (j, l) point l of mirror j , so that the field amplitude $A(x)$ is represented by the array $\{A_{j,l}\}_{j=1,2,\dots,N}^{l=1,2,\dots,M}$, giving a total of $N \times M$ points. The next step consists in choosing a finite-differences form of the integral appearing in the mechanical equations (8). We have found that, for stability purposes, an integration rule of the type

$$\int_{S_j} dx |A(x, t)|^2 \approx a \sum_{l=0}^{M+1} d_l |A_{j,l}(t)|^2, \quad (14)$$

where $A_{j,0} = A_{j-1,M}$ and $A_{j,M+1} = A_{j+1,1}$, is what works best, that is, we use a discrete representation of the integral over mirror j that includes the last point of the previous mirror and the first point of the next one. The weights satisfy the constrain $\sum_{l=0}^{M+1} d_l = 1$, and we have chosen a second order integration rule $\{d_l\}_{l=0,1,\dots,M+1} = \{1, 23, 24, 24, \dots, 24, 23, 1\}/24M$ which seems to provide very good convergence properties.

Following our previous work [7], and in order to identify the independent parameters of the model, we have defined the following dimensionless versions of the mechanical displacements and optical field,

$$z_j = \frac{4k_L}{T} q_j, \quad F = \frac{2}{\Omega_m} \sqrt{\frac{2\hbar k_c k_L a}{t_c m T}} A. \quad (15)$$

Combining this normalization with the discrete form (14) of the integral, and introducing dimensionless versions of time and space, $\tau = \gamma_c t$ and $\bar{x} = x/l_c$, respectively, we obtain the normalized equations

$$\frac{d^2 z_j}{d\tau^2} + \gamma \frac{dz_j}{d\tau} + \Omega^2 z_j = \quad (16a)$$

$$\rho^2 \Omega^2 \frac{l_c^2}{a^2} \sum_{\langle l \rangle_j} (z_l - z_j) + \Omega^2 \sum_{l=0}^{M+1} d_l |F_{j,l}|^2,$$

$$\partial_\tau F = (-1 + i\Delta + i\partial_{\bar{x}}^2 + iZ) F + E, \quad (16b)$$

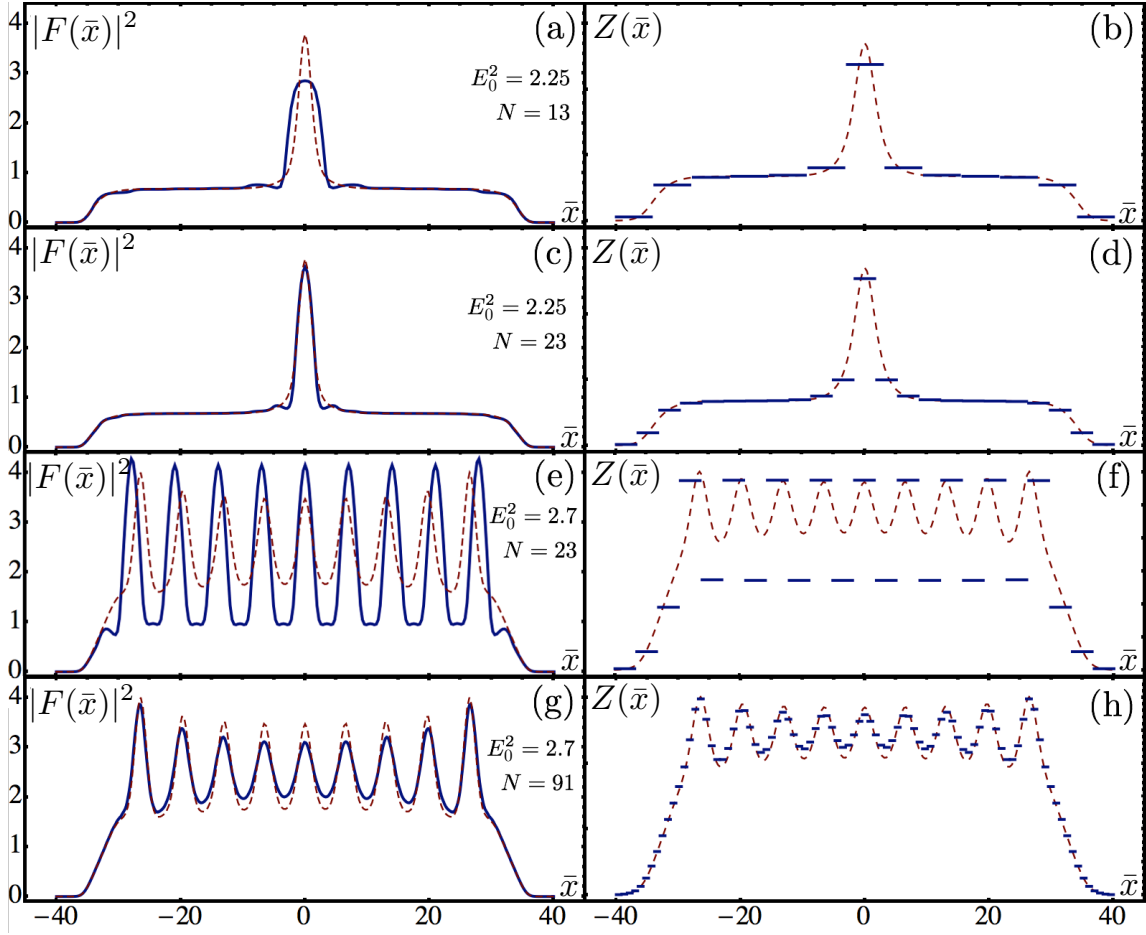


FIG. 2: Normalized field amplitude (squared) $|F(x)|^2$ and mechanical field $Q(x)$ in the steady-state as a function of the position, for a 1D system of finite size $x \in [-40l_c, 40l_c]$. The solutions have been obtained by numerical resolution of the discrete (solid blue) and continuous (dashed red) models described in the text, and under a top-hat illumination. In all cases $\gamma = 0.1$, $\Omega = 10$, $\Delta = -2.2$, and $\rho = 1.13$. The injection values E_0^2 have been chosen in the region where solitons (a-d) or periodic patterns (e-h) are expected from the continuous model. For the discrete model we consider N micro-mirrors as specified in the figure, so that their size is $a = 80l_c/N$.

with the normalized mechanical field written as $Z(\bar{x}) = \sum_j z_j w_j(l_c \bar{x})$. In these expressions we have rewritten $\kappa_\perp/m = v^2/a^2$ in terms of the effective rigidity parameter $\rho = v/\Omega_m l_c$ which, together with the detuning, was shown to control the appearance of dissipative structures in the continuous model [7]. We have also introduced normalized versions of other parameters, namely $\gamma = \gamma_m/\gamma_c$, $\Omega = \Omega_m/\gamma_m$, and $E = (2/\Omega_m)\sqrt{2\hbar k_c k_L a/t_c m T \mathcal{E}}$.

In order to simulate real experimental conditions, we have assumed a top-hat injection profile with finite width, modelled as a super-Gaussian $E(\bar{x}) = E_0 \exp(-\bar{x}^{20}/2\sigma_x^{20})$. In Fig. 2, we show in solid blue the stationary structures we have found in a spatial window $x \in [-40l_c, +40l_c]$ for different number of micromirrors N (whose size is then $a = 80l_c/N$), taking $M = 11$ field points per micromirror. We have chosen $\gamma = 0.1$, $\Omega = 10$, $\Delta = -2.2$, and $\rho = 1.13$, and studied the spatial structures for two values of the injection, $E_0^2 = 2.25$ and 2.7 (with $\sigma_x = 40$). For these parameters the continu-

ous limit predicts the appearance of, respectively, cavity solitons and periodic patterns [7]. In the figure we show in dashed red the corresponding structures found in the continuous limit for these same parameters. It is remarkable how cavity solitons are well captured even with a fairly small number of micromirrors. On the other hand, the restrictions on the number of micromirrors needed to see the periodic patterns are a bit tighter, because we need enough space to hold such an extended structure, but they can still be observed with not so many micromirrors, as we show below.

In Fig. 3 we illustrate what happens in the contrary limit, i.e., when the number of micromirrors is small, a limit that is easier to implement experimentally. We take $N = 7$ micromirrors and show how localized structures can be supported by a single micromirror. We are able to write and erase such structures by an additional Gaussian injection at the desired position, and hence they are equivalent to the cavity solitons present in the continuous

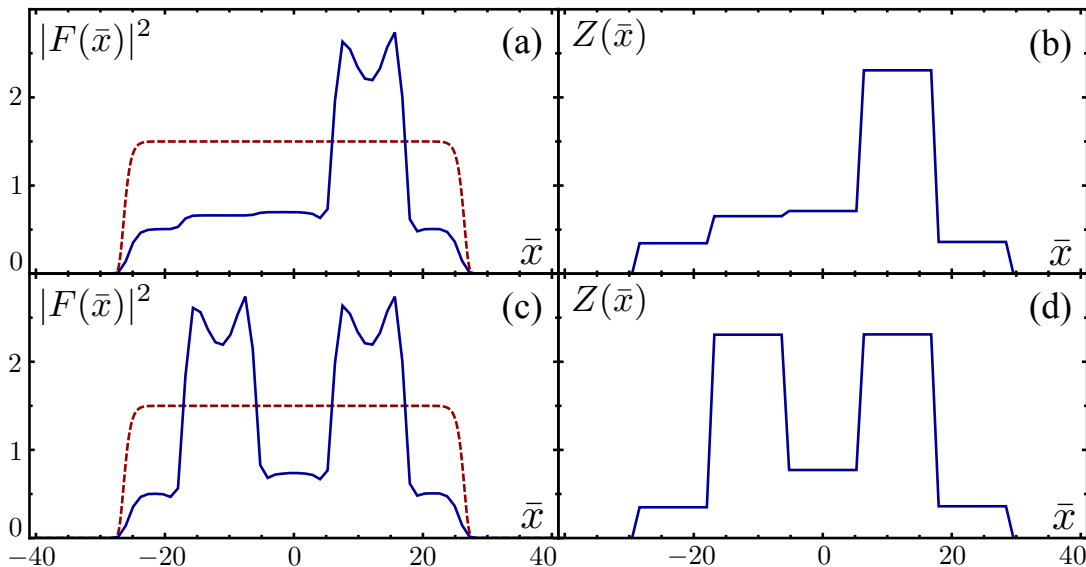


FIG. 3: Normalized field amplitude (squared) $|F(x)|^2$ and mechanical field $Q(x)$ in the steady-state as a function of the position, for a 1D system of finite size $x \in [-40l_c, 40l_c]$. Figs. (a) and (b) show a localized structure that has been written around position $\bar{x} = 12$; an additional localized structure has been written around $\bar{x} = -12$ in Figs. (c) and (d), which clearly does not disturb the previous structure. The basic shape of the injection (E^2) is represented in dashed red, on top of which a Gaussian profile with the proper width and position is initially fed in order to write the localized structures. The parameters in this simulation are $N = 7$, $\gamma = 0.1$, $\Omega = 10$, $\Delta = -2.2$, $\rho = 1.13$, $E_0 = \sqrt{1.5}$, and $\sigma_x = 23$.

case.

V. CONCLUSIONS

In this paper we have proposed an architecture for an optomechanical cavity that allows for the generation of dissipative structures. The device consists in an optomechanical cavity with an oscillating end-mirror consisting of an array of weakly-coupled micro-mirrors. This configuration fulfils the basic requirement necessary for optomechanical dissipative structures: the existence of a homogeneous mechanical transverse mode [7]. This proposal offers then an alternative to our previous one [7] that consisted in mounting a flexible mirror on a large aspect ratio frame (with dimensions $L_x \gg L_y$).

The model we are proposing coincides mathematically with that studied in Ref. [7] in the limit of large density of micro-mirrors, making all its analytical and numerical results applicable in such limit. We have numerically shown that this is also the case, more qualitatively, when the number of micro-mirrors is not very large, and have found that with a relatively small number of elements there exist solutions reminiscent of the continuous-case cavity solitons. More concretely, we have found that a discrete model consisting in $N \approx 20$ micro-mirrors with size $a \approx 4l_c$ is enough to observe localized structures exactly as predicted by the continuous limit model. Periodic patterns may require a larger number of micro-mirrors depending on their periodicity, but in any case they should still be well captured with a reasonable num-

ber of these (say $N < 100$). We hope that our work be useful in the experimental search of dissipative structures in optomechanical devices.

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VI. APPENDIX A: INTERPRETATION OF THE OPTICAL FIELD AMPLITUDE AND RADIATION PRESSURE

In the main text we wrote the electric field propagating to the right as

$$\mathbf{E}_+(z, \mathbf{r}, t) = i\mathbf{x}\mathcal{V}A_+(z, \mathbf{r}, t)e^{ik_L z - i\omega_L t} + \text{c.c.}, \quad (17)$$

where here we include the polarization of the electric field (defining the x direction with corresponding unit vector \mathbf{x}), which was omitted in the main text for simplicity. The aim of this section is to explain how the choice $\mathcal{V} = \sqrt{\hbar\omega_c/4\epsilon_0 L}$ characteristic of quantum optics allows us to give a simple interpretation to the amplitude A_+ (and similarly for A_- and A_{inj}), as well as writing the expression for the radiation pressure exerted by \mathbf{E}_+ in terms of this amplitude.

Let us first remind that, within the paraxial approximation, the magnetic field associated to (17) can be written as (\mathbf{y} is the unit vector in the y direction)

$$\mathbf{B}_+(z, \mathbf{r}, t) = i\mathbf{y}c^{-1}\mathcal{V}A_+(z, \mathbf{r}, t)e^{ik_L z - i\omega_L t} + \text{c.c.}; \quad (18)$$

the corresponding Poynting vector is then written as (\mathbf{z} is the unit vector in the z direction)

$$\mathbf{S}_+ = \frac{1}{\mu_0}\mathbf{E}_+ \times \mathbf{B}_+ = -\frac{\mathcal{V}^2 \mathbf{z}}{\mu_0 c} (A_+ e^{ik_L z - i\omega_L t} - \text{c.c.})^2, \quad (19)$$

whose magnitude averaged over an optical cycle

$$\begin{aligned} \langle S_+ \rangle|_{z=L} &= \frac{2\pi}{\omega_L} \int_{t-\pi/\omega_L}^{t+\pi/\omega_L} d\tau |\mathbf{S}_+(L, \mathbf{r}, \tau)| \\ &\simeq \frac{2\mathcal{V}^2}{\mu_0 c} |A(\mathbf{r}, t)|^2, \end{aligned} \quad (20)$$

provides the instantaneous measurable power impinging point \mathbf{r} of the mirror located at $z = L$ per unit area (irradiance). Note that we have made use of the slowly time-varying nature of the amplitude, and remember that we defined $A(\mathbf{r}, t) = A_+(L, \mathbf{r}, t)$ in the main text. Now it is customary in quantum optics to take $\mathcal{V} = \sqrt{\hbar\omega_c/4\epsilon_0 L}$ so that

$$|A(\mathbf{r}, t)|^2 = \frac{t_c \langle S_+ \rangle|_{z=L}}{\hbar\omega_c}, \quad (21)$$

can be interpreted as the number of photons per unit area which arrive to point \mathbf{r} of the mirror during a round-trip ($t_c = 2L/c$ is the cavity round-trip time). With this choice, the theory is quantized by interpreting this amplitude as an operator satisfying equal-time commutation relations $[\hat{A}(\mathbf{r}, t), \hat{A}^\dagger(\mathbf{r}', t)] = \delta(\mathbf{r} - \mathbf{r}')$ and $[\hat{A}(\mathbf{r}, t), \hat{A}(\mathbf{r}', t)] = 0$.

From the Poynting vector, we can get the radiation pressure exerted onto a point \mathbf{r} of the flexible mirror as $P(\mathbf{r}, t) = \langle S_+ \rangle|_{z=L}/c$; this is a quantity of fundamental relevance to our work, and in our case takes the particular expression

$$P(\mathbf{r}, t) = \frac{\hbar k_c}{t_c} |A(\mathbf{r}, t)|^2; \quad (22)$$

given our interpretation of $|A(\mathbf{r}, t)|^2$, this coincides precisely with the total momentum (momentum per photon \times number of photons) hitting point \mathbf{r} of the flexible mirror per unit time and area.

VII. APPENDIX B: DERIVATION OF THE LIGHT FIELD EQUATION

Here we derive equation (3) of the main text. To this aim we use the approach of references [7, 15], which consists in propagating the complex amplitudes $A_\pm(z, \mathbf{r}, t)$ along a full cavity round-trip. Assuming that they are

slowly varying in space and time, they satisfy the paraxial wave equation

$$(\partial_z \pm c^{-1}\partial_t) A_\pm = \pm \frac{i}{2k_L} \nabla_\perp^2 A_\pm. \quad (23)$$

Given the amplitude $A_+(z = L, \mathbf{r}, t)$, after reflection on the flexible mirror we get

$$A_-(L, \mathbf{r}, t) e^{-ik_L L} = -A_+(L, \mathbf{r}, t) e^{ik_L [L + 2Q(\mathbf{r}, t)]}, \quad (24)$$

where $Q(\mathbf{r}, t)$ represents the displacement of the mirror from its rest position ($Q = 0$ at rest). The amplitude $A_-(L, \mathbf{r}, t)$ propagates from $z = L$ to $z = 0$ giving rise to a new amplitude

$$A_- \left(0, \mathbf{r}, t + \frac{1}{2}t_c \right) = U_L A_-(L, \mathbf{r}, t), \quad (25)$$

where

$$U_L = \exp [i(L/2k_L)\nabla_\perp^2], \quad (26)$$

is the paraxial propagation operator in free space. After reflection onto the coupling mirror, a new amplitude

$$\begin{aligned} A_+ \left(0, \mathbf{r}, t + \frac{1}{2}t_c \right) &= -\sqrt{R} A_- \left(0, \mathbf{r}, t + \frac{1}{2}t_c \right) \\ &\quad + \sqrt{T} A_{\text{inj}} \left(0, \mathbf{r}, t + \frac{1}{2}t_c \right), \end{aligned} \quad (27)$$

is got, with R and T the reflectivity and transmissivity factors of the coupling mirror, respectively ($R + T = 1$ is assumed: lossless mirror). Finally, propagation from $z = 0$ to $z = L$ yields $A_+(L, \mathbf{r}, t + t_c) = U_L A_+(0, \mathbf{r}, t + \frac{1}{2}t_c)$. Adding all parts together one gets

$$\begin{aligned} A(\mathbf{r}, t + t_c) &= \sqrt{R} e^{2ik_L L} U_L^2 \exp [2ik_L Q(\mathbf{r}, t)] A(\mathbf{r}, t) \\ &\quad + \sqrt{T} A_{\text{inj}}(L, \mathbf{r}, t + t_c), \end{aligned} \quad (28)$$

where we used $U_L A_{\text{inj}}(0, \mathbf{r}, t + \frac{1}{2}t_c) = A_{\text{inj}}(L, \mathbf{r}, t + t_c)$. We now take into account that $R \rightarrow 1$ (equivalently, $T \rightarrow 0$) so that $\sqrt{R} = \sqrt{1 - T} \rightarrow 1 - T/2$. Next we assume that light is almost resonant with the cavity, specifically we impose that $2(\omega_L - \omega_c)L/c = \delta$ is of order T , where ω_c is the cavity longitudinal mode frequency (hence $\omega_c = m\pi c/L$, $m \in \mathbb{N}$) closest to ω_L , what allows approximating $\exp(2ik_L L) = \exp(2i\omega_L L/c) \approx 1 + i\delta$. We assume as well that $k_L Q(\mathbf{r}, t)$ is of order T (the mirror displacement/deformations are much smaller than the optical wavelength), so that $\exp [2ik_L Q(\mathbf{r}, t)] \approx 1 + 2ik_L Q(\mathbf{r}, t)$. Similarly we assume that the effect of diffraction is small (this implies that both mirrors must be sufficiently close each other, either physically or by means of lenses) so that we can expand $U_L^2 \approx 1 + i(L/k_L)\nabla_\perp^2$. All these assumptions imply that the overall variation of A between consecutive round-trips is very small and then one can approximate $\partial_t A$ by $[A(\mathbf{r}, t + t_c) - A(\mathbf{r}, t)] t_c^{-1}$. With all

these approximations we get, to the lowest nontrivial order,

$$\partial_t A(\mathbf{r}, t) = \gamma_c \left(-1 + i\Delta + i l_c^2 \nabla_\perp^2 + i \frac{4k_L}{T} Q \right) A + \gamma_c \mathcal{E}, \quad (29)$$

where all the parameters are defined in the main text; this is precisely Eq. (3), and it is the same light-field equation we derived in the linear-coupling model of Ref. [7].

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- [1] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
 - [2] K. Staliunas and V. J. Sánchez-Morcillo, *Transverse Patterns in Nonlinear Optical Resonators*, (Springer, Berlin, 2003).
 - [3] P. Mandel, *Theoretical problems in cavity nonlinear optics, Vol. 21* (Cambridge University Press, 2005).
 - [4] W.J. Firth, and C.O. Weiss, Opt. Photon. News **13** (2), 54 (2002).
 - [5] S. Barland et al., Nature **419**, 699 (2002).
 - [6] T. Ackemann, W. Firth, G.-L. Oppo, Adv. At. Mol. Opt. Phys. **57**, 323 (2009).
 - [7] J. Ruiz-Rivas, C. Navarrete-Benlloch, G. Patera, E. Roldán, and G.J. de Valcárcel, Phys. Rev. A (2015). Preprint arXiv:1212.1364
 - [8] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Rev. Mod. Phys. **86**, 1391 (2014).
 - [9] L. A. Lugiato and G. Grynberg, Europhys. Lett. **29**, 675 (1995).
 - [10] L. A. Lugiato and A. Gatti, Phys. Rev. Lett. **70**, 3868 (1993); A. Gatti and L. A. Lugiato, Phys. Rev. A **52**, 1675 (1995).
 - [11] M. Santagiustina, P. Colet, M. San Miguel, and D. Walgraef, Phys. Rev. Lett. **79**, 3633 (1997).
 - [12] I. Pérez-Arjona, E. Roldán, and G. J. de Valcárcel, Europhys. Lett. **74**, 247 (2006); Phys. Rev. A **75**, 063802 (2007).
 - [13] C. Navarrete-Benlloch, E. Roldán, and G. J. de Valcárcel, Phys. Rev. Lett. **100**, 203601 (2008).
 - [14] M. Vaupel, A. Maître, and C. Fabre, Phys. Rev. Lett. **83**, 5278 (1999).
 - [15] S. Kolpakov, A. Esteban-Martín, F. Silva, J. García, K. Staliunas, and G. J. de Valcárcel, Phys. Rev. Lett. **101**, 254101 (2008).